

Primitive Quantum BCH Codes over Finite Fields

Salah A. Aly, Andreas Klappenecker, Pradeep Kiran Sarvepalli

Department of Computer Science

Texas A&M University

College Station, TX 77843-3112, USA

Email: {salah,klappi,pradeep}@cs.tamu.edu

Abstract—An attractive feature of BCH codes is that one can infer valuable information from their design parameters (length, size of the finite field, and designed distance), such as bounds on the minimum distance and dimension of the code. In this paper, it is shown that one can also deduce from the design parameters whether or not a primitive, narrow-sense BCH contains its Euclidean or Hermitian dual code. This information is invaluable in the construction of quantum BCH codes. A new proof is provided for the dimension of BCH codes with small designed distance, and simple bounds on the minimum distance of such codes and their duals are derived as a consequence. These results allow us to derive the parameters of two families of primitive quantum BCH codes as a function of their design parameters.

I. INTRODUCTION

Let α denote a primitive element in the finite field \mathbf{F}_{q^m} . We set $n = q^m - 1$ and denote by δ an integer in the range $2 \leq \delta \leq n$. Recall that a cyclic code of length n over \mathbf{F}_q is called a primitive, narrow-sense BCH code with designed distance δ if its generator polynomial is of the form

$$g(x) = \prod_{z \in Z} (x - \alpha^z) \quad \text{with} \quad Z = C_1 \cup \dots \cup C_{\delta-1},$$

where $C_x = \{xq^k \bmod n \mid 0 \leq k < m\}$ denotes the q -ary cyclotomic coset of x modulo n . We refer to such a code as a $BCH(n, q; \delta)$ code, and call Z the defining set of the code. The basic properties of these classical codes are discussed, for example, in the books [9], [10], [11].

Given a classical BCH code, we can use one of the following well-known constructions to derive a quantum stabilizer code:

- 1) If there exists a classical linear $[n, k, d]_q$ code C such that $C^\perp \subseteq C$, then there exists an $[[n, 2k - n, \geq d]]_q$ stabilizer code that is pure to d . If the minimum distance of C^\perp exceeds d , then the quantum code is pure and has minimum distance d .
- 2) If there exists a classical linear $[n, k, d]_{q^2}$ code D such that $D^{\perp_h} \subseteq D$, then there exists an $[[n, 2k - n, \geq d]]_q$ stabilizer code that is pure to d . If the minimum distance of D^{\perp_h} exceeds d , then the quantum code is pure and has minimum distance d .

The orthogonality relations are defined in the *Notations* at the end of this section. Examples of certain binary quantum BCH codes have been given in [2], [5], [7], [13].

Our goal is to derive the parameters of the quantum stabilizer code as a function of their design parameters n , q , and δ

of the associated primitive, narrow-sense BCH code C . This entails the following tasks:

- a) Determine the design parameters for which $C^\perp \subseteq C$;
- b) determine the dimension of C ;
- c) bound the minimum weight in $C \setminus C^\perp$.

In case q is a perfect square, we would also like to answer the Hermitian versions of questions a) and c):

- a') Determine the design parameters for which $C^{\perp_h} \subseteq C$;
- c') bound the minimum weight in $C \setminus C^{\perp_h}$.

To put our work into perspective, we sketch our results and give a brief overview of related work.

Let C be a primitive, narrow-sense BCH code C of length $n = q^m - 1$, $m \geq 2$, over \mathbf{F}_q with designed distance δ .

To answer question a), we prove in Theorem 2 that $C^\perp \subseteq C$ holds if and only if $\delta \leq q^{\lceil m/2 \rceil} - 1 - (q - 2)[m \text{ odd}]$. The significance of this result is that allows one to identify all BCH codes that can be used in the quantum code construction 1). Fortunately, this question can be answered now without computations. Steane proved in [12] the special case $q = 2$, which is easier to show, since in this case there is no difference between even and odd m .

In Theorem 4, we answer question a') and show that $C^{\perp_h} \subseteq C$ if and only if $\delta \leq q^{(m + [m \text{ even}])/2} - 1 - (q - 2)[m \text{ even}]$, where we assume that q is a perfect square. This result allows us to determine all primitive, narrow-sense BCH codes that can be used in construction 2). We are not aware of any prior work concerning the Hermitian case.

In the binary case, an answer to question b) was given by MacWilliams and Sloane [11, Chapter 9, Corollary 8]. Apparently, Yue and Hu answered question b) in the case of small designed distances [15]. We give a new proof of this result in Theorem 7 and show that the dimension $k = n - m \lceil (\delta - 1)(1 - 1/q) \rceil$ for δ in the range $2 \leq \delta < q^{\lceil m/2 \rceil} + 1$. As a consequence of our answer to b), we obtain the dimensions of the quantum codes in constructions 1) and 2).

Finding the true minimum distance of BCH codes is an open problem for which a complete answer seems out of reach, see [3]. As a simple consequence of our answer to b), we obtain better bounds on the minimum distance for some BCH codes, and we derive simple bounds on the (Hermitian) dual distance of BCH codes with small designed distance, which partly answers c) and c').

In Section V, all these results are used to derive two families of quantum BCH codes. Impatient readers should now browse this section to get the bigger picture. Theorem 12 yields the result that one obtains using construction 1). The result of construction 2) is given in Theorem 13.

Notations. We denote the ring of integers by \mathbf{Z} and a finite field with q elements by \mathbf{F}_q . We follow Knuth and attribute to $[P(k)]$ the value 1 if the property $P(k)$ of the integer k is true, and 0 otherwise. For instance, we have $[k \text{ even}] = k - 1 \bmod 2$, but the left hand side seems more readable. If x and y are vectors in \mathbf{F}_q^n , then we write $x \perp y$ if and only if $x \cdot y = 0$. Similarly, if x and y are vectors in $\mathbf{F}_{q^2}^n$, then we write $x \perp_h y$ if and only if $x^q \cdot y = 0$.

II. EUCLIDEAN DUAL CODES

Recall that the Euclidean dual code C^\perp of a code $C \subseteq \mathbf{F}_q^n$ is given by $C^\perp = \{y \in \mathbf{F}_q^n \mid x \cdot y = 0 \text{ for all } x \in C\}$. Steane showed in [12] that a primitive binary BCH code of length $2^m - 1$ contains its dual if and only if its designed distance δ satisfies $\delta \leq 2^{\lceil m/2 \rceil} - 1$. In this section we derive a similar condition for nonbinary BCH codes.

Lemma 1: Suppose that $\gcd(n, q) = 1$. A cyclic code of length n over \mathbf{F}_q with defining set Z contains its Euclidean dual code if and only if $Z \cap Z^{-1} = \emptyset$, where Z^{-1} denotes the set $Z^{-1} = \{-z \bmod n \mid z \in Z\}$.

Proof: See, for instance, [6, Lemma 2] or [9, Theorem 4.4.11]. ■

Theorem 2: A primitive, narrow-sense BCH code of length $q^m - 1$, with $m \geq 2$, over the finite field \mathbf{F}_q contains its dual code if and only if its designed distance δ satisfies

$$\delta \leq \delta_{\max} = q^{\lceil m/2 \rceil} - 1 - (q - 2)[m \text{ odd}].$$

Proof: Let $n = q^m - 1$. The defining set Z of a primitive, narrow-sense BCH code C of designed distance δ is given by $Z = C_1 \cup C_2 \cdots \cup C_{\delta-1}$, where $C_x = \{xq^j \bmod n \mid j \in \mathbf{Z}\}$.

- 1) We will show that the code C cannot contain its dual code if the designed distance $\delta > \delta_{\max}$. Seeking a contradiction, we assume that the defining set Z contains the set $\{1, \dots, s\}$, where $s = \delta_{\max}$. By Lemma 1, it suffices to show that $Z \cap Z^{-1}$ is not empty. If m is even, then $s = q^{m/2} - 1$, and Z^{-1} contains the element $-sq^{m/2} \equiv q^{m/2} - 1 \equiv s \bmod n$, which means that $Z \cap Z^{-1} \neq \emptyset$; contradiction. If m is odd, then $s = q^{(m+1)/2} - q + 1$, and the element given by $-sq^{(m-1)/2} \equiv q^{(m+1)/2} - q^{(m-1)/2} - 1 \bmod n$ is contained in Z^{-1} . Since this element is less than s for $m \geq 3$, it is contained in Z , so $Z \cap Z^{-1} \neq \emptyset$; contradiction. Combining these two cases, we can conclude that $\delta \leq q^{\lceil m/2 \rceil} - 1 - (q - 2)[m \text{ odd}]$ for $m \geq 2$.
- 2) For the converse, we prove that if $\delta \leq \delta_{\max}$, then $Z \cap Z^{-1} = \emptyset$, which implies $C^\perp \subseteq C$ by Lemma 1. It suffices to show that $\min C_{-x} \geq \delta_{\max}$ for any coset C_x in Z . Since $1 \leq x < \delta_{\max} \leq q^{\lceil m/2 \rceil} - 1$, we can write x as a q -ary integer of the form $x = x_0 + x_1q + \cdots + x_{m-1}q^{m-1}$ with $0 \leq x_i < q$, and $x_i = 0$ for $i \geq \lceil m/2 \rceil$.

If $\bar{y} = n - x$, then $\bar{y} = \bar{y}_0 + \bar{y}_1q + \cdots + \bar{y}_{m-1}q^{m-1} = \sum_{i=0}^{m-1} (q - 1 - x_i)q^i$. Set $y = \min C_{-x}$. We note that y is a conjugate of \bar{y} . Thus, the digits of y are obtained by cyclically shifting the digits of \bar{y} .

- 3a) First we consider the case when m is even. Then the q -ary expansion of x has at least $m/2$ zero digits. Therefore, at least $m/2$ of the \bar{y}_i are equal to $q - 1$. Thus, $y \geq \sum_{i=0}^{m/2-1} (q - 1)q^i = q^{m/2} - 1 = \delta_{\max}$.
- 3b) If m is odd, then as $1 \leq x < q^{(m+1)/2} - q + 1$, we have $m > 1$ and $\bar{y} = \bar{y}_0 + \bar{y}_1q + \cdots + (\bar{y}_{(m-1)/2})q^{(m-1)/2} + (q - 1)q^{(m+1)/2} + \cdots + (q - 1)q^{m-1}$. For $0 \leq j \leq (m - 1)/2$, we observe that $xq^j < n$, and since $\bar{y}q^j \equiv -xq^j \bmod n$, $\bar{y}q^j = n - xq^j \geq q^m - 1 - (q^{(m+1)/2} - q)q^{(m-1)/2} = q^{(m+1)/2} - 1 \geq \delta_{\max}$. For $(m + 1)/2 \leq j \leq m - 1$, we find that

$$\begin{aligned} \bar{y}q^j \bmod n &= \bar{y}_{m-j} + \cdots + \bar{y}_{(m-1)/2}q^{j-(m+1)/2} \\ &\quad + (q - 1)q^{j-(m-1)/2} + \cdots + (q - 1)q^{j-1} \\ &\quad + \bar{y}_0q^j + \cdots + \bar{y}_{m-j-1}q^{m-1}, \\ &\geq (q^{(m-1)/2} - 1)q^{j-(m-1)/2} + \bar{y}_0 + \cdots \\ &\quad + \bar{y}_{(m-1)/2}, \\ &\geq q^{(m+1)/2} - q + 1 = \delta_{\max}, \end{aligned}$$

where $\bar{y}_0 + \cdots + \bar{y}_{(m-1)/2} \geq 1$ because $x < q^{(m+1)/2} - q + 1$. Hence $y = \min\{\bar{y}q^j \mid j \in \mathbf{Z}\} \geq \delta_{\max}$ when m is odd.

Therefore a primitive BCH code contains its dual if and only if $\delta \leq \delta_{\max}$, for $m \geq 2$. ■

III. HERMITIAN DUAL CODES

If the cardinality of the field is a perfect square, then we can define another type of orthogonality relation for codes. Recall that if the code C is a subspace of the vector space $\mathbf{F}_{q^2}^n$, then its Hermitian dual code C^{\perp_h} is given by $C^{\perp_h} = \{y \in \mathbf{F}_{q^2}^n \mid y^q \cdot x = 0 \text{ for all } x \in C\}$, where $y^q = (y_1^q, \dots, y_n^q)$ denotes the conjugate of the vector $y = (y_1, \dots, y_n)$. The goal of this section is to establish when a primitive, narrow-sense BCH code contains its Hermitian dual code.

Lemma 3: Assume that $\gcd(n, q) = 1$. A cyclic code of length n over \mathbf{F}_{q^2} with defining set Z contains its Hermitian dual code if and only if $Z \cap Z^{-q} = \emptyset$, where $Z^{-q} = \{-qz \bmod n \mid z \in Z\}$.

Proof: Let $N = \{0, 1, \dots, n - 1\}$. If $g(z) = \prod_{x \in Z} (z - \alpha^x)$ is the generator polynomial of a cyclic code C , then $h^\dagger(z) = \prod_{x \in N \setminus Z} (z - \alpha^{-qx})$ is the generator polynomial of C^{\perp_h} . Thus, $C^{\perp_h} \subseteq C$ if and only if $g(z)$ divides $h^\dagger(z)$. The latter condition is equivalent to $Z \subseteq \{-qx \mid x \in N \setminus Z\}$, which can also be expressed as $Z \cap Z^{-q} = \emptyset$. ■

Theorem 4: A primitive, narrow-sense BCH code of length $q^{2m} - 1$ over \mathbf{F}_{q^2} , where $m \neq 2$, contains its Hermitian dual code if and only if its designed distance δ satisfies

$$\delta \leq \delta_{\max} = q^{m + [m \text{ even}]} - 1 - (q^2 - 2)[m \text{ even}].$$

Proof: Let $n = q^{2m} - 1$. Recall that the defining set Z of a primitive, narrow-sense BCH code C over the finite field

\mathbb{F}_{q^2} with designed distance δ is given by $Z = C_1 \cup \dots \cup C_{\delta-1}$ with $C_x = \{xq^{2j} \bmod n \mid j \in \mathbb{Z}\}$.

- 1) We will show that the code C cannot contain its Hermitian dual code if the designed distance $\delta > \delta_{\max}$. Seeking a contradiction, we assume that the defining set Z contains $\{1, \dots, s\}$, where $s = \delta_{\max}$. By Lemma 3, it suffices to show that $Z \cap Z^{-q}$ is not empty. If m is odd, then $s = q^m - 1$. Notice that $n - qsq^{2(m-1)/2} = q^m - 1 = s$, which means that $s \in Z \cap Z^{-q}$, and this contradicts our assumption that this set is empty. If m is even, then $s = q^{m+1} - q^2 + 1$. We note that $n - qsq^{m-2} = q^{m+1} - q^{m-1} - 1 < s = q^{m+1} - q^2 + 1$, for $m > 2$. It follows that $q^{m+1} - q^{m-1} - 1 \in Z \cap Z^{-q}$, contradicting our assumption that this set is empty. Combining the two cases, we can conclude that s must be smaller than the value $q^{m+[m \text{ even}]} - 1 - (q^2 - 2)[m \text{ even}]$.
- 2) For the converse, we show that if $\delta < \delta_{\max}$, then $Z \cap Z^{-q} = \emptyset$, which implies $C^{\perp_h} \subseteq C$ thanks to Lemma 3. It suffices to show that $\min\{n - qC_x\} \geq \delta_{\max}$ or, equivalently, that $\max qC_x \leq n - \delta_{\max}$ holds for $1 \leq x \leq \delta - 1$.
- 3) If m is odd, then the q -ary expansion of x is of the form $x = x_0 + x_1q + \dots + x_{m-1}q^{m-1}$, with $x_i = 0$, for $m \leq i \leq 2m - 1$ as $x < q^m - 1$. So at least m of the x_i are equal to zero, which implies $\max qC_x < q^{2m} - 1 - (q^m - 1) = n - \delta_{\max}$.
- 4) Let m be even and qxq^{2j} be the q^2 -ary conjugates of qx . Since $x < q^{m+1} - q^2 + 1$, $x = x_0 + x_1q + \dots + x_mq^m$ and at least one of the $x_i \leq q - 2$. If $0 \leq 2j \leq m - 2$, then $qxq^{2j} \leq q(q^{m+1} - q^2)q^{m-2} = q^{2m} - q^{m+1} = n - q^{m+1} + 1 < n - \delta_{\max}$. If $2j = m$, then $qxq^m = x_{m-1} + x_mq + 0 \cdot q^2 + \dots + 0 \cdot q^m + x_0q^{m+1} \dots + x_{m-2}q^{2m-1}$. We note that there occurs a consecutive string of $m - 1$ zeros and because one of the $x_i \leq q - 2$, we have $qxq^{2j} < n - q^2(q^{m-1} - 1) - 1 \leq n - \delta_{\max}$. For $m + 2 \leq 2j \leq 2m - 2$, we see that $qxq^{2j} < n - q^4(q^{m-1} - 1) < n - \delta_{\max}$.

Thus we can conclude that the primitive BCH codes contain their Hermitian duals when $\delta \leq q^{m+[m \text{ even}]} - 1 - (q^2 - 2)[m \text{ even}]$. ■

IV. DIMENSION AND MINIMUM DISTANCE

In this section we determine the dimension of primitive, narrow-sense BCH codes of length n with small designed distance. Furthermore, we derive bounds on the minimum distance of such codes and their duals.

A. Dimension

First, we make some simple observations about cyclotomic cosets that are essential in our proof.

Lemma 5: If q be a power of a prime, m a positive integer and $n = q^m - 1$, then all q -ary cyclotomic cosets $C_x = \{xq^\ell \bmod n \mid \ell \in \mathbb{Z}\}$ with x in the range $1 \leq x < q^{\lceil m/2 \rceil} + 1$ have cardinality $|C_x| = m$.

Proof: Seeking a contradiction, we assume that $|C_x| < m$. If $m = 1$, then C_x would have to be the empty set, which is impossible. If $m > 1$, then $|C_x| < m$ implies that there

must exist an integer j in the range $1 \leq j < m$ such that j divides m and $xq^j \equiv x \bmod n$. In other words, $q^m - 1$ divides $x(q^j - 1)$; hence, $x \geq (q^m - 1)/(q^j - 1)$.

If m is even, then $j \leq m/2$; thus, $x \geq q^{m/2} + 1$. If m is odd, then $j \leq m/3$ and it follows that $x \geq (q^m - 1)/(q^{m/3} - 1)$, and it is easy to see that the latter term is larger than $q^{\lceil m/2 \rceil} + 1$. In both cases this contradicts our assumption that $1 \leq x \leq q^{\lceil m/2 \rceil}$; hence $|C_x| = m$. ■

Lemma 6: Let q be a power of a prime, m a positive integer, and $n = q^m - 1$. Let x and y be integers in the range $1 \leq x, y < q^{\lceil m/2 \rceil} + 1$ such that $x, y \not\equiv 0 \bmod q$. If $x \neq y$, then the q -ary cosets of x and y modulo n are disjoint, i.e., $C_x \neq C_y$.

Proof: Seeking a contradiction, we assume that $C_x = C_y$. This assumption implies that $y \equiv xq^\ell \bmod n$ for some integer ℓ in the range $1 \leq \ell < m$.

If $xq^\ell < n$, then $xq^\ell \equiv 0 \bmod q$; this contradicts our assumption $y \not\equiv 0 \bmod q$, so we must have $xq^\ell \geq n$. It follows from the range of x that ℓ must be at least $\lfloor m/2 \rfloor$.

If $\ell = \lfloor m/2 \rfloor$, then we cannot find an admissible x within the given range such that $y \equiv xq^{\lfloor m/2 \rfloor} \bmod n$. Indeed, it follows from the inequality $xq^{\lfloor m/2 \rfloor} \geq n$ that $x \geq q^{\lceil m/2 \rceil}$, so x must equal $q^{\lceil m/2 \rceil}$, but that contradicts $x \not\equiv 0 \bmod q$. Therefore, ℓ must exceed $\lfloor m/2 \rfloor$.

Let us write x as a q -ary number $x = x_0 + x_1q + \dots + x_{m-1}q^{m-1}$, with $0 \leq x_i < q$. Note that $x_0 \neq 0$ because $x \not\equiv 0 \bmod q$. If $\lfloor m/2 \rfloor < \ell < m$, then xq^ℓ is congruent to $y_0 = x_{m-\ell} + \dots + x_{m-1}q^{\ell-1} + x_0q^\ell + \dots + x_{m-\ell-1}q^{m-1}$ modulo n . We observe that $y_0 \geq x_0q^\ell \geq q^{\lceil m/2 \rceil}$. Since $y \not\equiv 0 \bmod q$, it follows that $y = y_0 \geq q^{\lceil m/2 \rceil} + 1$, contradicting the assumed range of y . ■

The previous two observations about cyclotomic cosets allow us to derive a closed form for the dimension of a primitive BCH code. This result generalizes binary case [11, Corollary 9.8, page 263]. See also [14] which gives estimates on the dimension of BCH codes among other things.

Theorem 7: A primitive, narrow-sense BCH code of length $q^m - 1$ over \mathbb{F}_q with designed distance δ in the range $2 \leq \delta \leq q^{\lceil m/2 \rceil} + 1$ has dimension

$$k = q^m - 1 - m[(\delta - 1)(1 - 1/q)]. \quad (1)$$

Proof: The defining set of the code is of the form $Z = C_1 \cup C_2 \cup \dots \cup C_{\delta-1}$, a union of at most $\delta - 1$ consecutive cyclotomic cosets. However, when $1 \leq x \leq \delta - 1$ is a multiple of q , then $C_{x/q} = C_x$. Therefore, the number of cosets is reduced by $\lfloor (\delta - 1)/q \rfloor$. By Lemma 6, if $x, y \not\equiv 0 \bmod q$ and $x \neq y$, then the cosets C_x and C_y are disjoint. Thus, Z is the union of $(\delta - 1) - \lfloor (\delta - 1)/q \rfloor = \lceil (\delta - 1)(1 - 1/q) \rceil$ distinct cyclotomic cosets. By Lemma 5 all these cosets have cardinality m . Therefore, the degree of the generator polynomial is $m[(\delta - 1)(1 - 1/q)]$, which proves our claim about the dimension of the code. ■

If we exceed the range of the designed distance in the hypothesis of the previous theorem, then our dimension formula (1) is no longer valid, as our next example illustrates.

Example 8: Consider a primitive, narrow-sense BCH code of length $n = 4^2 - 1 = 15$ over \mathbb{F}_4 . If we choose the designed

distance $\delta = 6 > 4^1 + 1$, then the resulting code has dimension $k = 8$, because the defining set Z is given by

$$Z = C_1 \cup C_2 \cup \dots \cup C_5 = \{1, 4\} \cup \{2, 8\} \cup \{3, 12\} \cup \{5\}.$$

The dimension formula (1) yields $4^2 - 1 - 2\lceil(6-1)(1-1/4)\rceil = 7$, so the formula does not extend beyond the range of designed distances given in Theorem 7.

B. Distance Bounds

The true minimum distance d_{\min} of a primitive BCH code over \mathbb{F}_q with designed distance δ is bounded by $\delta \leq d_{\min} \leq q\delta - 1$, see [11, p. 261]. If we apply the Farr bound (essentially the sphere packing bound) using the dimension given in Theorem 7, then we obtain:

Corollary 9: If C is primitive, narrow-sense BCH code of length $q^m - 1$ over \mathbb{F}_q with designed distance δ in the range $2 \leq \delta \leq q^{\lceil m/2 \rceil} + 1$ such that

$$\sum_{i=0}^{\lfloor (\delta+1)/2 \rfloor} \binom{q^m - 1}{i} (q-1)^i > q^{m\lceil(\delta-1)(1-1/q)\rceil}, \quad (2)$$

then C has minimum distance $d = \delta$ or $\delta + 1$; if, furthermore, $\delta \equiv 0 \pmod q$, then $d = \delta + 1$.

Proof: Seeking a contradiction, we assume that the minimum distance d of the code satisfies $d \geq \delta + 2$. We know from Theorem 7 that the dimension of the code is $k = q^m - 1 - m\lceil(\delta-1)(1-1/q)\rceil$. If we substitute this value of k into the sphere-packing bound

$$q^k \sum_{i=0}^{\lfloor (d-1)/2 \rfloor} \binom{q^m - 1}{i} (q-1)^i \leq q^n,$$

then we obtain

$$\sum_{i=0}^{\lfloor (\delta+1)/2 \rfloor} \binom{q^m - 1}{i} (q-1)^i \leq \sum_{i=0}^{\lfloor (d-1)/2 \rfloor} \binom{q^m - 1}{i} (q-1)^i \leq q^{m\lceil(\delta-1)(1-1/q)\rceil},$$

but this contradicts condition (2); hence, $\delta \leq d \leq \delta + 1$.

If $\delta \equiv 0 \pmod q$, then the cyclotomic coset C_δ is contained in the defining set Z of the code because $C_\delta = C_{\delta/q}$. Thus, the BCH bound implies that the minimum distance must be at least $\delta + 1$. ■

Lemma 10: Suppose that C is a primitive, narrow-sense BCH code of length $n = q^m - 1$ over \mathbb{F}_q with designed distance $2 \leq \delta \leq \delta_{\max} = q^{\lceil m/2 \rceil} - 1 - (q-2)\lceil m \text{ odd} \rceil$, then the dual distance $d^\perp \geq \delta_{\max} + 1$.

Proof: Let $N = \{0, 1, \dots, n-1\}$ and Z_δ be the defining set of C . We know that $Z_{\delta_{\max}} \supseteq Z_\delta \supset \{1, \dots, \delta-1\}$. Therefore $N \setminus Z_{\delta_{\max}} \subseteq N \setminus Z_\delta$. Further, we know that $Z \cap Z^{-1} = \emptyset$ if $2 \leq \delta \leq \delta_{\max}$ from Lemma 1 and Theorem 2. Therefore, $Z_{\delta_{\max}}^{-1} \subseteq N \setminus Z_{\delta_{\max}} \subseteq N \setminus Z_\delta$.

Let T_δ be the defining set of the dual code. Then $T_\delta = (N \setminus Z_\delta)^{-1} \supseteq Z_{\delta_{\max}}$. Moreover $\{0\} \in N \setminus Z_\delta$ and therefore T_δ . Thus there are at least δ_{\max} consecutive roots in T_δ . Thus the dual distance $d^\perp \geq \delta_{\max} + 1$. ■

Lemma 11: Suppose that C is a primitive, narrow-sense BCH code of length $n = q^{2m} - 1$ over \mathbb{F}_{q^2} with designed distance $2 \leq \delta \leq \delta_{\max} = q^{m+\lceil m \text{ even} \rceil} - 1 - (q^2-2)\lceil m \text{ even} \rceil$, then the dual distance $d^\perp \geq \delta_{\max} + 1$.

Proof: The proof is analogous to the one of Lemma 10; just keep in mind that the defining set Z_δ is invariant under multiplication by q^2 modulo n . ■

V. FAMILIES OF QUANTUM CODES

We use the results of the previous sections to prove the existence of quantum stabilizer codes.

Theorem 12: If q is a power of a prime, and m and δ are integers such that $m \geq 2$ and $2 \leq \delta \leq \delta_{\max} = q^{\lceil m/2 \rceil} - 1 - (q-2)\lceil m \text{ odd} \rceil$, then there exists a quantum stabilizer code Q with parameters

$$[[q^m - 1, q^m - 1 - 2m\lceil(\delta-1)(1-1/q)\rceil, d_Q \geq \delta]]_q$$

that is pure up to $\delta_{\max} + 1$. If $BCH(n, q; \delta)$ has true minimum distance d , and $d \leq \delta_{\max}$, then Q is a pure quantum code with minimum distance $d_Q = d$.

Proof: Theorem 7 and 2 imply that there exists a classical BCH code with parameters $[q^m - 1, q^m - 1 - m\lceil(\delta-1)(1-1/q)\rceil, \geq \delta]_q$ which contains its dual code. An $[n, k, d]_q$ code that contains its dual code implies the existence of the quantum code with parameters $[[n, 2k - n, \geq d]]_q$ by the CSS construction, see [7], [5]. By Lemma 10, the dual distance exceeds δ_{\max} ; the statement about the purity and minimum distance is an immediate consequence. ■

Theorem 13: If q is a power of a prime, m is a positive integer, and δ is an integer in the range $2 \leq \delta \leq q^m - 1$, then there exists a quantum code Q with parameters

$$[[q^{2m} - 1, q^{2m} - 1 - 2m\lceil(\delta-1)(1-1/q^2)\rceil, d_Q \geq \delta]]_q$$

that is pure up to $\delta_{\max} + 1$, where $\delta_{\max} = q^{m+\lceil m \text{ even} \rceil} - 1 - (q^2-2)\lceil m \text{ even} \rceil$. If $BCH(n, q^2; \delta)$ has true minimum distance d , with $d \leq \delta_{\max}$, then Q is a pure quantum code of minimum distance $d_Q = d$.

Proof: It follows from Theorems 7 and 4 that there exists a primitive, narrow-sense $[q^{2m} - 1, q^{2m} - 1 - m\lceil(\delta-1)(1-1/q^2)\rceil, \geq \delta]_{q^2}$ BCH code that contains its Hermitian dual code. Recall that if a classical $[n, k, d]_{q^2}$ code C exists that contains its Hermitian dual code, then there exists an $[[n, 2k - n, \geq d]]_q$ quantum code that is pure up to d , see [1]; this proves our claim. By Lemma 11, the Hermitian dual distance exceeds δ_{\max} , which implies the last statement of the claim. ■

VI. CONCLUSIONS

We have investigated primitive, narrow-sense BCH codes in this note. We were able to characterize when primitive, narrow-sense BCH codes contain their Euclidean and Hermitian dual codes, and this allowed us to derive two series of quantum stabilizer codes. These results make it possible to construct more families of quantum BCH codes as shown by Cohen, Encheva, and Litsyn in [4], since the BCH codes are nested and are amenable to the Steane enlargement technique [12].

From a practical point of view, it is interesting that efficient encoding and decoding algorithms are known for cyclic binary quantum stabilizer codes, see [8].

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